Int. J. Solids Structures Vol. 28, No. 9, pp. 1139–1154, 1991
Printed in Great Britain.

0020–7683/91 $3.00+ .00

PLASTIC BUCKLING OF METAL MATRIX LAMINATED PLATES

M. PALEY and J. ABOUDI
Department of Solid Mechanics, Materials and Structures, Faculty of Engineering, Tel-Aviv University, Ramat-Aviv, Tel-Aviv 69978, Israel

(*Received* 17 *August* 1990; *in revised form* 2 *December* 1990)

Abstract—A method is proposed for the determination of plastic bifurcation buckling load of metal matrix composite plates. The metallic matrix behavior is described by an elastic–viscoplastic constitutive law, while the fibers are assumed to be either elastic or elastic–viscoplastic material. The approach is based on the load level and history dependent instantaneous effective properties of the inelastic plate, which are established by a micromechanical analysis. An incremental procedure is developed in which a buckling condition has to be established and its fulfilment must be checked at each increment. The method is applied for the prediction of the plastic buckling of boron/aluminum composite plates in various situations, by employing the classical and higher order shear deformation plate theories.

1. INTRODUCTION

The determination of the buckling load of perfectly elastic composite plates has received considerable attention [see Leissa (1985, 1987) for extensive reviews]. Most of the papers covered in these reviews utilize the classical theory of plates. The use of a first order theory in the study of bifurcation buckling of elastic plates can be found in Whitney and Leissa (1969) and Whitney (1987). Analyses of composite plate buckling by employing higher order theories can be found in the recent review by Kapania and Raciti (1989).

The stability analysis of homogeneous inelastic plates is much more complicated than the perfectly elastic case, due to the inherent nonlinearity of the material constitutive law. The study of bifurcation buckling can be performed by employing total deformation plasticity (e.g., Bijlaard, 1949) or incremental theory (e.g., Hendelman and Prager, 1948). For extensive reviews of this subject, see Hutchinson (1974) and Bushnell (1982). Recently, the influence of material rate sensitivity on the elastic–plastic buckling of structures has been investigated by Tvergaard (1985), Bodner *et al.* (1991), and Paley and Aboudi (1991b).

The theory of uniqueness and bifurcation in time-independent elastic–plastic materials was given by Hill (1958). It was shown by Tvergaard (1989) that Hill's bifurcation condition can be satisfied for elastic–viscoplastic materials only at the elastic buckling load. This is due to the high strain-rates that occur at the instant of buckling. In practice, imperfections and inertial effects reduce the instantaneous changes of buckling strain rates (Bodner *et al.*, 1991). The use of strain-rates in the pre-buckled state in an appropriate buckling condition of a viscoplastic material provides lower bounds to the actual buckling loads.

The prediction of plastic buckling of metal matrix composite structures is difficult due to the complicated interactions between the inelastic matrix and reinforcement. In previous investigations, the inelastic effects of the matrix were neglected by assuming perfectly elastic behavior for all loading levels. Obviously, this is a restriction since a metal matrix composite deforms plastically at the early stages of loading. For example, in a boron/aluminum composite the aluminum matrix yields at a strain of about 0.002, which is far away from the ultimate strain of the composite. Thus, an efficient utilization of the metal matrix composite up to failure must take into account the plastic flow of the matrix. The plastic buckling analysis of a metal matrix composite structure requires the knowledge of its instantaneous stiffnesses which depend on the history and loading level. The overall instantaneous properties of the inelastic composite can be determined by a suitable micromechanical analysis from the known material parameters of fiber and matrix. In the framework of the classical incremental plasticity theory, bounds on the overall instantaneous elastoplastic moduli of periodic composites were determined by Accorci and Nemat-Nasser (1986) and Teply and Dvorak (1988).

Explicit expressions of the effective instantaneous properties of metal matrix composites were recently given by Paley and Aboudi (1991a). The derivation of the instantaneous properties was based on the micromechanical method of cells (Aboudi, 1989) in which the metal matrix is assumed to behave as an elastic–viscoplastic material reinforced by elastic or elastic–viscoplastic fibers. It was shown by Paley and Aboudi (1991a) that at any stage of loading, the current stiffnesses of the inelastic fiber and matrix material can be employed to generate, in conjunction with the micromechanical analysis, the overall stiffness tensor of the unidirectional composite. The instantaneous stiffnesses of metal matrix composite laminates were obtained by using the standard lamination theory.

In the present paper, the previous micromechanical method for the derivation of instantaneous properties of metal matrix composites is utilized for the bifurcation buckling analysis of composite plates, where the matrix material could be elastic–viscoplastic. In particular, the Bodner and Partom (1975) model is employed in the applications given in this paper. The method is based on the development of governing buckling equations of the composite inelastic plate, which are expressed in terms of the velocity field, and involve the instantaneous properties of the laminated plate. Consequently, the derived buckling condition, which is obtained from the governing equations, depends on the loading level and its history. Due to the existing plastic behavior of an inelastic phase, the method of obtaining the critical buckling load level is incremental. At each load increment the fulfilment of the derived buckling condition is examined. If the condition is satisfied, then the buckling load level of the composite plate has been achieved; otherwise the incremental procedure is continued.

The method is illustrated for various types of boron/aluminum composite plates. Both the classical and higher order shear deformation plate theories are employed. The effects of fiber volume ratio, plate thickness, material rate sensitivity, temperature, ply orientation and total number of layers on the plastic buckling load of the plate are studied. Comparisons with the corresponding buckling loads obtained by disregarding the plasticity effects of the metal matrix are presented.

2. MATERIAL STIFFNESS REDUCTION DUE TO PLASTIC FLOW

2.1. *Homogeneous viscoplastic materials*

Consider a homogeneous isotropic elastic–viscoplastic material subjected to a slowly increasing external loading. At the first stage of loading, the material behavior is essentially elastic and is characterized by its elastic stiffnesses. At a later stage, plastic deformation develops, leading to anisotropic behavior of the material, which is described by its instantaneous reduced stiffness properties. This stiffness reduction, which is caused by the plastic flow of the material, can be determined at any instant of the loading by the following considerations.

The total strain-*rate* tensor of the material is assumed to be a sum of two components: reversible (elastic) η^E_{ij} and irreversible (plastic) η^P_{ij}, i.e.,

$$\eta_{ij} = \eta^E_{ij} + \eta^P_{ij}. \tag{1}$$

The stress-*rate* tensor τ_{ij} is related to the elastic part of the total strain-*rate* by the generalized Hooke's law (small deformations are assumed):

$$\tau_{ij} = C_{ijkl}\eta^E_{kl} \tag{2}$$

where C_{ijkl} is the fourth order elastic stiffness tensor of the material, and the summation convention is employed on repeated italic letters. The plastic components of the strain-*rate* are controlled by the Prandtl–Reuss flow law as follows:

$$\eta_{ij}^{p} = \Lambda s_{ij} \tag{3}$$

where $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ is the deviatoric part of the stress tensor σ_{ij}, δ_{ij} is the Kronecker delta, and Λ is the flow rule function of the adopted viscoplastic constitutive model which describes the inelastic behavior of the material.

The following constitutive relation for the initially isotropic homogeneous viscoplastic material can be established (Paley and Aboudi, 1991b):

$$\tau_{ij} = C_{ijkl}^{VP}\eta_{kl} \tag{4}$$

where the instantaneous *strain-rate dependent* viscoplastic moduli C_{ijkl}^{VP} of the material are defined by

$$C_{ijkl}^{VP} = C_{ijkl} - g s_{ij} s_{kl}. \tag{5}$$

In (5), the scalar g is defined by the following equation:

$$g = \frac{2\mu}{s_{mn}s_{mn}} \frac{s_{rt}\eta_{rt}^{p}}{s_{pq}\eta_{pq}} \tag{6}$$

where μ is the elastic rigidity of the material. Equations (4)–(6) were established after various manipulations by employing the flow rule (3) in conjunction with eqns (1) and (2). It should be noted that these relations were established without the use of the yield function concept. In this paper, the superscript $(\ )^{VP}$ is used to denote the various current properties which characterize the viscoplastic material (and structure) that vary with loading history. It can be readily seen that the reduction of material stiffness, caused by plastic flow, is a function of the history and the current state of loading.

The instantaneous constitutive relation (4) was previously employed by the authors to investigate the bifurcation buckling of viscoplastic plates (Paley and Aboudi, 1991b). This was performed by utilizing, at any stage of loading, the current value of C_{ijkl}^{VP} of the material at the buckling condition of the plate, which was derived from the governing equations in conjunction with the specified boundary conditions.

It can be observed that the rate-dependent C_{ijkl}^{VP} are similar to the form of the rate-independent instantaneous moduli used by Hill in the derivation of his bifurcation condition [see, for example, eqn (3.4) of Tvergaard (1989)]. One can readily see from eqn (6) that $g = 0$ when the material behavior is elastic, whereas g is positive when plastic flow takes place. The latter follows from the fact that g is given in terms of the ratio of the plastic work rate ($s_{rt}\eta_{rt}^{p}$) to the rate of the total work of deformation ($s_{pq}\eta_{pq}$) which does not include volume changes.

In the present case of a viscoplastic material the strain-rate at the pre-buckled state is used in computing C_{ijkl}^{VP}. At the instant of buckling, however, high strain-rates develop in the material. Consequently, the predicted values of critical loads are lower bounds of the actual values. Thus the obtained bifurcation results are approximate, representing predictions for an analogous time-independent material. Similar arguments were given by Bodner *et al.* (1991).

By establishing a constitutive relation similar to eqn (4), for the unidirectional lamina of a composite plate, it will be shown in the sequel that an analogous procedure can be adopted for the analysis of the bifurcation buckling of metal matrix laminated plates.

2.2. *Unidirectional metal matrix composites*

Let us consider a unidirectional metal matrix composite material which consists of perfectly elastic fibers reinforcing an elastic–viscoplastic matrix. The fibers are oriented in the x_1' direction of a Cartesian coordinate system (x_1', x_2', x_3'). For transversely isotropic fibers (x_2'–x_3' is the plane of isotropy), the material stiffness tensor of the fibers is given by

$$C_{ijkl}^{(f)} = \lambda^{(f)}\delta_{ij}\delta_{kl} + \mu^{(f)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}) + \alpha^{(f)}(\delta_{ij}\delta_{1k}\delta_{1l} + \delta_{kl}\delta_{1i}\delta_{1j})$$
$$+ \beta^{(f)}(\delta_{ik}\delta_{1j}\delta_{1l} + \delta_{jk}\delta_{1i}\delta_{1l} + \delta_{il}\delta_{1j}\delta_{1k} + \delta_{jl}\delta_{1i}\delta_{1k}) + \gamma^{(f)}\delta_{1i}\delta_{1j}\delta_{1k}\delta_{1l} \quad (7)$$

where $\lambda^{(f)}$, $\mu^{(f)}$, $\alpha^{(f)}$, $\beta^{(f)}$ and $\gamma^{(f)}$ are five independent constants that characterize the fiber material, and can be easily related to its corresponding engineering constants. The above tensor can be rewritten in a more familiar form using a contracted notation.

$$\mathbf{C}^{(f)} = \begin{bmatrix} C_{11}^{(f)} & C_{12}^{(f)} & C_{12}^{(f)} & 0 & 0 & 0 \\ & C_{22}^{(f)} & C_{23}^{(f)} & 0 & 0 & 0 \\ & & C_{22}^{(f)} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{22}^{(f)} - C_{23}^{(f)}) & 0 & 0 \\ & & & & C_{66}^{(f)} & 0 \\ \text{symm.} & & & & & C_{66}^{(f)} \end{bmatrix}.$$

The elastic stiffness of the isotropic matrix phase is given by

$$C_{ijkl}^{(m)} = \lambda^{(m)}\delta_{ij}\delta_{kl} + \mu^{(m)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (8)$$

where $\mu^{(m)}$ and $\lambda^{(m)}$ are Lamé constants of the material.

Employing eqns (4)–(6) for the fiber and matrix phases, one obtains the following viscoplastic constitutive equations for the q-phase ($q = f, m$)

$$\tau_{ij}^{(q)} = C_{ijkl}^{VP(q)}\eta_{kl}^{(q)} \quad (9)$$

where

$$C_{ijkl}^{VP(q)} = C_{ijkl}^{(q)} - g^{(q)}s_{ij}^{(q)}s_{kl}^{(q)} \quad (10)$$

with

$$g^{(q)} = \frac{2\mu^{(q)}}{s_{mn}^{(q)}s_{mn}^{(q)}}\,\frac{s_{rt}^{(q)}\eta_{rt}^{P(q)}}{s_{ij}^{(q)}\eta_{ij}^{(q)}}. \quad (11)$$

For perfectly elastic fibers the plastic strain rate $\eta_{ij}^{P(f)} \equiv 0$, implying that $g^{(f)} = 0$ and $C_{ijkl}^{VP(f)} = C_{ijkl}^{(f)}$. It is readily seen that the present derivation also includes the more general case of elastic–viscoplastic fibers. In such a case eqns (9)–(11) can be used subject to the requirement that the inelastic fibers are initially isotropic.

The overall instantaneous properties of the unidirectional fiber-reinforced material can be determined by a suitable micromechanical analysis in which the detailed interaction between fiber and matrix phases is considered. Such a micromechanical analysis was developed by Paley and Aboudi (1991a) by employing the method of cells of Aboudi (1989). This composite model is based on the analysis of a repeating cell, and leads to the prediction of the overall behavior of various types of composites from the known properties of the fiber and matrix materials.

It was shown by Paley and Aboudi (1991a) that the knowledge of the instantaneous properties of the inelastic fiber and matrix phases provides, in conjunction with micromechanical analysis of the method of cells, the overall instantaneous stiffness of the unidirectional metal matrix composite. Consequently, from the knowledge of $C_{ijkl}^{VP(q)}$ ($q = f, m$) at any stage of loading, one can determine the effective instantaneous behavior of the composite in the form

$$\bar{\tau}_{ij} = C_{ijkl}^{*VP}\bar{\eta}_{kl} \quad (12)$$

where $\bar{\tau}_{ij}$ and $\bar{\eta}_{ij}$ are the average stress-rate and strain-rate tensors, and C^{*VP}_{ijkl} is the fourth order tensor of the effective instantaneous reduced stiffnesses of the composite. Thus, the composite constitutive equation (12) is a generalization of relation (4), which was established for a homogeneous viscoplastic material. The explicit expression of C^{*VP}_{ijkl} can be found in Paley and Aboudi (1991a). It should be emphasized that the tensor C^{*VP}_{ijkl} depends on the level and history of loading. It is worth mentioning that a constitutive equation of a similar form was also established by Paley and Aboudi (1991a) for composite laminates in which each lamina is a unidirectional metal matrix composite.

3. PLASTIC BUCKLING OF LAMINATED PLATES

In this section, we will utilize the previously discussed instantaneous constitutive relation (12) for inelastic unidirectional composites to establish the governing equations for plastic buckling of metal matrix laminates. To this end, consider a rectangular laminated plate which consists of several laminae. The middle plane of the laminate coincides with the x–y plane of the Cartesian coordinate system (xyz). The x and y axes are parallel to the plate edges, and the origin of the system is taken at a corner. The dimensions of the plate are: a and b in the x and y directions, respectively, and thickness h in the z direction. When it is convenient, we will use the following alternative notation for the axes: $x_1 = x$, $x_2 = y$ and $x_3 = z$. Each layer of the laminate is a unidirectional metal matrix composite which is assumed to be in the plane-stress state. Consequently, the constitutive equation (12) of the single lamina in its material coordinate system ($x'_1 x'_2 x'_3$) can be rewritten as

$$\bar{\tau}_i = Q^{*VP}_{ij}\bar{\eta}_j \quad (i, j = 1, 2, 4, 5, 6) \tag{13}$$

where $\bar{\tau}_1 = \bar{\tau}_{x'_1x'_1}$, $\bar{\tau}_2 = \bar{\tau}_{x'_2x'_2}$, $\bar{\tau}_4 = \bar{\tau}_{x'_2x'_3}$, $\bar{\tau}_5 = \bar{\tau}_{x'_1x'_3}$, $\bar{\tau}_6 = \bar{\tau}_{x'_1x'_2}$, and $\bar{\eta}_1 = \bar{\eta}_{x'_1x'_1}$, $\bar{\eta}_2 = \bar{\eta}_{x'_2x'_2}$, $\bar{\eta}_4 = 2\bar{\eta}_{x'_2x'_3}$, $\bar{\eta}_5 = 2\bar{\eta}_{x'_1x'_3}$, $\bar{\eta}_6 = 2\bar{\eta}_{x'_1x'_2}$. The nonzero elements of Q^{*VP}_{ij} are obtained from C^{*VP}_{ij} of the corresponding layer [where C^{*VP}_{ij} is the instantaneous effective material stiffness tensor C^{*VP}_{ijkl}, eqn (12), which is rewritten in the contracted notation] as follows

$$Q^{*VP}_{ij} = C^{*VP}_{ij} - C^{*VP}_{i3}C^{*VP}_{3j}/C^{*VP}_{33} \quad (i, j = 1, 2, 6)$$

and

$$Q^{*VP}_{ij} = C^{*VP}_{ij} \quad (i, j = 4, 5). \tag{14}$$

The plane-stress instantaneous stiffness $\hat{Q}^{VP}_{ij}$ of the lamina, expressed in the global coordinate system (xyz), is obtained by the appropriate transformation of the above defined Q^{*VP}_{ij}. In these plate coordinates, eqn (13) becomes

$$\hat{\tau}_i = \hat{Q}^{*VP}_{ij}\hat{\eta}_j \quad (i, j = 1, 2, 4, 5, 6) \tag{15}$$

where $\hat{\tau}_i$ and $\hat{\eta}_i$ are the average stress-rates and strain-rates in the lamina expressed in the plate coordinate system. As usual, the average strain-rates are taken to be equal to the strain-rates in the plate. Assuming small changes of displacement field during each load increment, the total strain-rates in the plate in the global coordinates become

$$\hat{\eta}_{ij} = \tfrac{1}{2}(\partial \dot{u}_i/\partial x_j + \partial \dot{u}_j/\partial x_i) \tag{16}$$

where $\dot{u}_i$ are components of the assumed velocity field associated with adopted laminated plate theory. Both higher order and classical plate theories will be employed to establish the governing equations of the plastic bifurcation buckling of the considered laminated plate.

3.1. *Higher order plate theory*

The adopted higher order shear deformation laminate theory (HSDT) is based on a third order expansion of the displacement field (Reddy and Phan, 1985). In the present case of inelastic behavior, the following third order expansion of the velocity components is assumed:

$$\dot{u}_1 = \dot{u}(x,y) + z\dot{\psi}_x - \frac{4}{3h^2}z^3(\dot{\psi}_x + \partial\dot{w}/\partial x)$$
$$\dot{u}_2 = \dot{v}(x,y) + z\dot{\psi}_y - \frac{4}{3h^2}z^3(\dot{\psi}_y + \partial\dot{w}/\partial y)$$
$$\dot{u}_3 = \dot{w}(x,y) \quad (17)$$

where $\dot{u}$, $\dot{v}$ and $\dot{w}$ are the x, y and z components of velocity of middle plane points; $\dot{\psi}_x$ and $\dot{\psi}_y$ are rotation rates of transverse normals about the y and x axes, respectively.

Substitution of the above defined velocities into (16) leads to the following field of total strain-rates in the laminated plate:

$$\dot{\eta}_{11} = \eta_1^0 + z\kappa_1^0 + z^3\kappa_1^2, \quad \dot{\eta}_{22} = \eta_2^0 + z\kappa_2^0 + z^3\kappa_2^2, \quad \dot{\eta}_{12} = \eta_6^0 + z\kappa_6^0/2 + z^3\kappa_6^2/2,$$
$$\dot{\eta}_{23} = \eta_4^0 + z^2\kappa_4^2/2, \quad \dot{\eta}_{13} = \eta_5^0 + z^2\kappa_5^2/2, \quad (18)$$

where the following definitions are employed:

$$\eta_1^0 = \partial\dot{u}/\partial x, \quad \eta_2^0 = \partial\dot{v}/\partial y, \quad \eta_6^0 = \tfrac{1}{2}(\partial\dot{u}/\partial y + \partial\dot{v}/\partial x),$$
$$\eta_4^0 = \tfrac{1}{2}(\dot{\psi}_y + \partial\dot{w}/\partial y), \quad \eta_5^0 = \tfrac{1}{2}(\dot{\psi}_x + \partial\dot{w}/\partial x), \quad (19)$$

and

$$\kappa_1^0 = \partial\dot{\psi}_x/\partial x, \quad \kappa_2^0 = \partial\dot{\psi}_y/\partial y, \quad \kappa_6^0 = \partial\dot{\psi}_x/\partial y + \partial\dot{\psi}_y/\partial x,$$
$$\kappa_1^2 = -(4/3h^2)(\partial\dot{\psi}_x/\partial x + \partial^2\dot{w}/\partial x^2),$$
$$\kappa_2^2 = -(4/3h^2)(\partial\dot{\psi}_y/\partial y + \partial^2\dot{w}/\partial y^2),$$
$$\kappa_6^2 = -(4/3h^2)(\partial\dot{\psi}_x/\partial y + \partial\dot{\psi}_y/\partial x + 2\partial^2\dot{w}/\partial x\,\partial y),$$
$$\kappa_4^2 = -(4/h^2)(\dot{\psi}_y + \partial\dot{w}/\partial y), \quad \kappa_5^2 = -(4/h^2)(\dot{\psi}_x + \partial\dot{w}/\partial x). \quad (20)$$

The in-plane stress resultants are defined by

$$N_i = \int_{-h/2}^{h/2} \bar{\sigma}_i \, dz \quad (i = 1,2,6) \quad (21)$$

where $\bar{\sigma}_i$ are the stress components referred to the global coordinate system. The stress-rate resultants are defined by

$$(\dot{N}_i, \dot{M}_i, \dot{P}_i) = \int_{-h/2}^{h/2} \dot{\bar{\tau}}_i(1, z, z^3)\, dz \quad (i = 1,2,6)$$

and

$$(\dot{Q}_i, \dot{R}_i) = \int_{-h/2}^{h/2} \dot{\tau}_i(1, z^2)\, dz \quad (i = 4, 5); \tag{22}$$

as previously defined, $\dot{\tau}_1 = \dot{\tau}_{xx}$, $\dot{\tau}_2 = \dot{\tau}_{yy}$, $\dot{\tau}_4 = \dot{\tau}_{yz}$, $\dot{\tau}_5 = \dot{\tau}_{xz}$, $\dot{\tau}_6 = \dot{\tau}_{xy}$ are the components of the stress-rate tensor in the plate coordinate system, i.e., $\dot{\tau}_{ij} = \partial\hat{\sigma}_{ij}/\partial t$.

The equilibrium equations of the plate in terms of the stress-rate resultants become

$$\begin{aligned}
&\partial\dot{Q}_4/\partial x + \partial\dot{Q}_5/\partial y + (\partial/\partial x)(N_x\, \partial\dot{w}/\partial x) + (\partial/\partial y)(N_y\, \partial\dot{w}/\partial y) \\
&\qquad - (4/h^2)(\partial\dot{R}_4/\partial x + \partial\dot{R}_5/\partial y) + (4/3h^2)(\partial^2\dot{P}_1/\partial x^2 + 2\partial^2\dot{P}_6/\partial x\,\partial y + \partial^2\dot{P}_2/\partial y^2) = 0 \\
&\partial\dot{M}_1/\partial x + \partial\dot{M}_6/\partial y - \dot{Q}_4 + (4/h^2)\dot{R}_4 - (4/3h^2)(\partial\dot{P}_1/\partial x + \partial\dot{P}_6/\partial y) = 0 \\
&\partial\dot{M}_6/\partial x + \partial\dot{M}_2/\partial y - \dot{Q}_5 + (4/h^2)\dot{R}_5 - (4/3h^2)(\partial\dot{P}_6/\partial x + \partial\dot{P}_2/\partial y) = 0.
\end{aligned} \tag{23}$$

Here, N_x and N_y are the in-plane loads in the x and y directions, respectively, which are acting at time t. It should be noted that eqns (23) are meaningful during the buckling state where the rate of deflection of the plate becomes nonzero. In this instant, the out-of-plane rate resultants (22) become nonzero as well. The in-plane resultants N_i ($i = 1, 2, 6$) in eqn (21), on the other hand, are assumed to be constants during the instant of buckling, i.e., $\dot{N}_i = 0$.

The instantaneous stiffnesses of the plate are defined by

$$(A_{ij}^{VP}, B_{ij}^{VP}, D_{ij}^{VP}, E_{ij}^{VP}, F_{ij}^{VP}, H_{ij}^{VP}) = \int_{-h/2}^{h/2} \hat{Q}_{ij}^{VP}(1, z, z^2, z^3, z^4, z^6)\, dz \quad (i, j = 1, 2, 6)$$

$$(A_{ij}^{VP}, D_{ij}^{VP}, F_{ij}^{VP}) = \int_{-h/2}^{h/2} \hat{Q}_{ij}^{VP}(1, z^2, z^4)\, dz \quad (i, j = 4, 5). \tag{24}$$

Substituting eqns (22), in conjunction with (15) and (18)–(20), into (23) and using the definitions (24), we obtain the following constitutive equations for the plate:

$$(\dot{N}_i, \dot{M}_i, \dot{P}_i) = (A_{ij}^{VP}, B_{ij}^{VP}, E_{ij}^{VP})\eta_j^0 + (B_{ij}^{VP}, D_{ij}^{VP}, F_{ij}^{VP})\kappa_j^0 + (E_{ij}^{VP}, F_{ij}^{VP}, H_{ij}^{VP})\kappa_j^2 \quad (1, j = 1, 2, 6)$$

and

$$(\dot{Q}_i, \dot{R}_i) = (A_{ij}^{VP}, D_{ij}^{VP})\eta_j^0 + (D_{ij}^{VP}, F_{ij}^{VP})\kappa_j^2 \quad (i, j = 4, 5). \tag{25}$$

The present approach for the determination of plastic buckling load is illustrated for two types of composite plates as follows:

(1) symmetric cross-ply laminated plate;
(2) anti-symmetric angle-ply laminated plate.

The plates are initially at rest and subjected to a uniform, biaxial pressure. The following boundary conditions are assumed:

$$\begin{aligned}
\dot{w}(x, 0) &= \dot{w}(x, b) = \dot{w}(0, y) = \dot{w}(a, y) = 0, \\
\dot{P}_2(x, 0) &= \dot{P}_2(x, b) = \dot{P}_1(0, y) = \dot{P}_1(a, y) = 0, \\
\dot{M}_2(x, 0) &= \dot{M}_2(x, b) = \dot{M}_1(0, y) = \dot{M}_1(a, y) = 0, \\
\dot{\psi}_x(x, 0) &= \dot{\psi}_x(x, b) = \dot{\psi}_y(0, y) = \dot{\psi}_y(a, y) = 0.
\end{aligned} \tag{26}$$

For cross-ply laminates:

$$\dot{u}(x,0)=\dot{u}(x,b)=\dot{v}(0,y)=\dot{v}(a,y)=0,$$
$$\dot{N}_y(x,0)=\dot{N}_y(x,b)=\dot{N}_x(0,y)=\dot{N}_x(a,y)=0. \tag{27}$$

The boundary conditions for anti-symmetric angle-ply laminates are

$$\dot{v}(x,0)=\dot{v}(x,b)=\dot{u}(0,y)=\dot{u}(a,y)=0,$$
$$\dot{N}_{xy}(x,0)=\dot{N}_{xy}(x,b)=\dot{N}_{xy}(0,y)=\dot{N}_{xy}(a,y)=0. \tag{28}$$

These boundary conditions are exactly satisfied by

$$\dot{w}=\sum_{m,n=1}^{\infty} W_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y,$$
$$\dot{\psi}_x=\sum_{m,n=1}^{\infty} X_{mn}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y,\quad \dot{\psi}_y=\sum_{m,n=1}^{\infty} Y_{mn}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y. \tag{29}$$

For cross-ply laminates:

$$\dot{u}=\sum_{m,n=1}^{\infty} U_{mn}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y,\quad \dot{v}=\sum_{m,n=1}^{\infty} V_{mn}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y. \tag{30}$$

For anti-symmetric angle-ply laminates:

$$\dot{u}=\sum_{m,n=1}^{\infty} U_{mn}\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y,\quad \dot{v}=\sum_{m,n=1}^{\infty} V_{mn}\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y. \tag{31}$$

Following Reddy and Phan (1985), let us substitute (25), in conjunction with (19), (20) and (24), and using (29) and (30) for cross-ply, or (29) and (31) for angle-ply laminates, into (23) leads to the following five homogeneous equations for each set of numbers m and n:

$$[K_{ij}^{VP(mn)}-G_{ij}^{(mn)}]\Delta_j=0\quad (i,j=1,\ldots,5) \tag{32}$$

where $(\Delta_1,\ldots,\Delta_5)=(U_{mn},V_{mn},W_{mn},X_{mn},Y_{mn})$. The coefficients $K_{ij}^{VP(mn)}$ are mode-dependent linear combinations of the instantaneous stiffnesses of the inelastic plate and can be identified with the combinations of elastic stiffnesses denoted by C_{ij} in the Appendix of Reddy and Phan (1985). To this end, the elastic stiffnesses $(A_{ij}, B_{ij}, D_{ij},\ldots)$ there should be replaced by the corresponding instantaneous stiffnesses $(A_{ij}^{VP}, B_{ij}^{VP}, D_{ij}^{VP},\ldots)$. For example, the expression for $K_{11}^{VP(mn)}$ would therefore be given by $(m\pi/a)^2A_{11}^{VP}+(n\pi/b)^2A_{66}^{VP}$. The nonzero element of $G_{ij}^{(mn)}$ is given by

$$G_{33}^{(mn)}=N_x(m\pi/a)^2+N_y(n\pi/b)^2.$$

In order to obtain a nontrivial solution to (32), singularity of the 5×5 matrix of coefficients must be required. This leads to the following buckling condition

$$N_x(m\pi/a)^2+N_y(n\pi/b)^2=\frac{\det(\mathbf{K}^{VP(mn)})}{\det(\mathbf{\Psi}^{VP(mn)})} \tag{33}$$

where $\mathbf{K}^{VP(mn)}$ is a 5×5 matrix formed by the coefficients $K_{ij}^{VP(mn)}$, and $\mathbf{\Psi}^{VP(mn)}$ is a 4×4 matrix obtained from $\mathbf{K}^{VP(mn)}$ by deleting the row $i=3$ and the column $j=3$. From (33), one can see that the bifurcation buckling of the plate occurs when the combination of axial loads $N_x(m\pi/a)^2+N_y(n\pi/b)^2$ for any mode is equal to the above expression of the instantaneous stiffnesses of the laminate (given by the right-hand side of the equation). In the process of the determination of plastic buckling, the loads have to be applied progressively in an incremental manner. At each load increment the buckling condition, eqn (33), has to be checked to verify whether it is satisfied. In the case of a perfectly elastic composite plate, where the instantaneous properties are constants, eqn (33) readily provides the buckling load in terms of the laminated plate elastic properties (Reddy and Phan, 1985).

An important observation concerning the plastic buckling of inelastic plates is in order. Although the inelastic plate is initially anisotropic, additional anisotropy is developed due to the plastic flow of the inelastic phases (e.g., the metallic matrix). This follows from the fact that the instantaneous effective stiffnesses of the plate, which describe its anisotropy, vary with loading history. This implies that if buckling of the inelastic plate takes place in accordance with (33) at a certain mode (mn), the corresponding elastic buckling of the same plate (obtained by disregarding the plasticity effects) might occur at a different mode. Thus, the inclusion of plasticity effects of the constituents in the buckling analysis might change the load level and mode of buckling.

3.2. *Classical plate theory*

The following displacement rate field, associated with the classical laminated plate theory (CPT), is assumed:

$$\begin{aligned} \dot{u}_1 &= \dot{u}(x, y) - z\, \partial\dot{w}/\partial x \\ \dot{u}_2 &= \dot{v}(x, y) - z\, \partial\dot{w}/\partial y \\ \dot{u}_3 &= \dot{w}(x, y). \end{aligned} \tag{34}$$

In the framework of this theory, we will consider the following two types of uniformly, biaxially compressed laminated plates:

(a) anti-symmetrical angle-ply plate;
(b) orthotropic plate (unidirectional single-layer or symmetric cross-ply laminate).

In the first case of an angle-ply laminate, the bifurcation buckling of the plate is governed by the following set of equilibrium equations:

$$L_{ij}^{VP} u_j = 0 \quad (i, j = 1, 2, 3). \tag{35}$$

In the above equation, the differential operators L_{ij}^{VP} are obtained from the well known classical theory of laminated plates. The plate is initially at rest and the following boundary conditions are chosen:

$$\begin{aligned} \dot{w}(x, 0) &= \dot{w}(x, b) = \dot{w}(0, y) = \dot{w}(a, y) = 0, \\ \dot{v}(x, 0) &= \dot{v}(x, b) = \dot{u}(0, y) = \dot{u}(a, y) = 0, \\ \dot{M}_y(x, 0) &= \dot{M}_y(x, b) = \dot{M}_x(0, y) = \dot{M}_x(a, y) = 0, \\ \dot{N}_{xy}(x, 0) &= \dot{N}_{xy}(x, b) = \dot{N}_{xy}(0, y) = \dot{N}_{xy}(a, y) = 0. \end{aligned} \tag{36}$$

The solution is

$$\begin{aligned} \dot{u} &= \sum_{m,n=1}^{\infty} U_{mn} \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y, \\ \dot{v} &= \sum_{m,n=1}^{\infty} V_{mn} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y, \\ \dot{w} &= \sum_{m,n=1}^{\infty} W_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y. \end{aligned} \tag{37}$$

Substituting (37), in conjunction with (22), into (35), and requiring singularity of the matrix of coefficients of the resulting equations, we obtain the following buckling condition for the considered angle-ply laminate (Whitney, 1987):

$$N_x(m\pi/a)^2+N_y(n\pi/b)^2 = D_{11}^{VP}m^4+2(D_{12}^{VP}+2D_{66}^{VP})m^2n^2(a/b)^2+D_{22}^{VP}n^4(a/b)^4$$
$$-m[B_{16}^{VP}m^2+3B_{26}^{VP}n^2(a/b)^2]\frac{J_2^{VP(mn)}}{J_1^{VP(mn)}}-n(a/b)[3B_{16}^{VP}m^2+B_{26}^{VP}n^2(a/b)^2]\frac{J_3^{VP(mn)}}{J_1^{VP(mn)}} \quad (38)$$

where

$$J_1^{VP(mn)} = (A_{11}^{VP}m^2+A_{66}^{VP}n^2(a/b)^2)(A_{66}^{VP}m^2+A_{22}^{VP}n^2(a/b)^2)-m^2n^2(a/b)^2(A_{12}^{VP}+A_{66}^{VP})^2$$
$$J_2^{VP(mn)} = (A_{11}^{VP}m^2+A_{66}^{VP}n^2(a/b)^2)(B_{16}^{VP}m^2+3B_{26}^{VP}n^2(a/b)^2)$$
$$-3m^2n^2(a/b)^2(A_{12}^{VP}+A_{66}^{VP})^2(B_{16}^{VP}m^2+B_{26}^{VP}n^2(a/b)^2)$$
$$J_3^{VP(mn)} = (A_{66}^{VP}m^2+A_{22}^{VP}n^2(a/b)^2)(3B_{16}^{VP}m^2+B_{26}^{VP}n^2(a/b)^2)$$
$$-m^2n^2(a/b)^2(A_{12}^{VP}+A_{66}^{VP})^2(B_{16}^{VP}m^2+3B_{26}^{VP}n^2(a/b)^2).$$

In the case of a symmetrically laminated (all $B_{ij}=0$), orthotropic plate ($D_{16}=D_{26}=0$) that is simply supported, (35) is simplified to the following single equation:

$$D_{11}^{VP}\,\partial^4\dot{w}/\partial x^4+2(D_{12}^{VP}+2D_{66}^{VP})\,\partial^4\dot{w}/\partial x^2\,\partial y^2+D_{22}^{VP}\,\partial^4\dot{w}/\partial y^4$$
$$= N_x\,\partial^2\dot{w}/\partial x^2+2N_{xy}\,\partial^2\dot{w}/\partial x\,\partial y+N_y\,\partial^2\dot{w}/\partial y^2. \quad (39)$$

For biaxial load, the buckling condition becomes

$$N_x(m\pi/a)^2+N_y(n\pi/b)^2 = D_{11}^{VP}m^4+2(D_{12}^{VP}+2D_{66}^{VP})m^2n^2(a/b)^2+D_{22}^{VP}n^4(a/b)^4. \quad (40)$$

Having obtained a buckling condition of a metal matrix laminated plate [e.g. eqn (35), eqn (38), or eqn (40)] in terms of the instantaneous properties of the inelastic plate, we can proceed and determine the bifurcation buckling load level according to the following incremental procedure.

(a) At a given load level the stress resultants and the strain-rate field are known.

(b) These quantities determine the stresses and strain-rates of the single lamina.

(c) The determined fields of the lamina are used, in conjunction with the micromechanical analysis, to obtain the instantaneous effective stiffnesses C_{ijkl}^{*VP} of the unidirectional ply [eqn (12)].

(d) The instantaneous plate stiffnesses [defined by eqn (24)] can be readily determined from the computed C_{ijkl}^{*VP} of the laminae.

(e) The use of the instantaneous plate stiffnesses in a buckling condition determines whether buckling of the plate occurs at the present load level. If the buckling condition is not satisfied with the current plate stiffnesses, the load is modified and the above procedure is repeated.

The present method of the determination of plastic bifurcation buckling can be applied for various types of inelastic laminated plates, subjected to various boundary conditions. If an analytical, exact solution to the governing equations is not possible, approximate methods (e.g., numerical) can be employed in conjunction with the proposed methodology.

4. APPLICATIONS

The proposed method of bifurcation buckling analysis of metal matrix laminated plates is illustrated for the prediction of the critical load of a boron/aluminum composite system. The boron fibers are assumed to behave as a perfectly elastic material, whereas the aluminum matrix is represented as an elastic–viscoplastic work-hardening material. The aluminum alloy 2024-T4 is an almost rate-insensitive material at room temperature, but its rate sensitivity significantly increases at elevated temperatures. In this paper, the inelastic response of the matrix material is described by Bodner and Partom's (1975) unified visco-plasticity theory. This theory does not assume the existence of a yield condition, which eliminates the need to specify loading or unloading criteria, and the same equations can be directly used in all stages of loading and unloading. According to these equations plastic deformation always exists, but it is negligibly small when the material behavior should be essentially elastic.

The flow rule function Λ in eqn (3) is given, according to this theory, as follows:

$$\Lambda = D_0 \exp(-\tilde{n}[Z^2/(3J_2)]^n)/J_2^{1/2} \tag{41}$$

where $\tilde{n} = \frac{1}{2}(n+1)/n$ and $J_2 = \frac{1}{2}s_{ij}s_{ij}$ is the second invariant of the stress deviator; D_0 and n are inelastic material parameters, and Z is a state variable that represents the hardened state of the material with respect to resistance to plastic flow. In the case of isotropic hardening, the evolution law of this variable is given by

$$\dot{Z} = m(Z_1 - Z)\dot{W}_P/Z_0 \tag{42}$$

where Z_0, Z_1 and m are additional inelastic parameters of the material. Note that the plastic work W_P, whose rate is $\dot{W}_P = \sigma_{ij}\dot{\eta}_{ij}^{(P)}$, is taken as the measure of hardening. The physical significance of the above inelastic constants is as follows. The parameter D_0 is the limiting strain rate, Z_0 is related to the "yield stress" of a uniaxial stress–strain curve, and Z_1 is proportional to the ultimate stress. The material constant m determines the rate of work hardening, and the rate sensitivity is controlled by the temperature-dependent parameter n.

In Table 1, the material parameters of the 2024-T4 aluminum alloy at various temperatures are given in the framework of Bodner Partom theory. The table presents the elastic moduli, E and ν, Young's modulus and Poisson's ratio, respectively, and five inelastic parameters, namely: D_0, Z_0, Z_1, m and n. It should be noted that higher temperatures are associated with lower values of n, which results in an increase of the material rate sensitivity. In Fig. 1, the stress–strain response of the material is given for various values of temperature.

The boron fibers are considered to be perfectly elastic and isotropic with a Young's modulus of 400 GPa and Poisson's ratio of 0.3.

The present approach is applied to investigate square ($a = b$) laminated cross-ply and angle-ply plates. In all cases the thickness of each ply is 1 mm. The effects of the following parameters on the buckling of the plate will be considered: length-to-thickness ratio (a/h), fiber volume ratio (v_f), applied strain-rate ($\dot{\eta}_{xx}$), angle of lamination (θ), number of plies (k) and temperature (T). Some of these effects are examined in the framework of CPT and HSDT.

The inelastic buckling of the composite plates is analyzed by the application of a uniaxial compressive stress loading $\hat{\sigma}_{xx}$ while controlling the strain-rate value $\hat{\dot{\eta}}_{xx}$. For convenience the stresses and strain-rates are shown in the figures as positive.

Table 1. Material constants of an aluminum alloy (2024-T4)

Temperature (°C)	E (GPa)	ν	D_0 (s^{-1})	Z_0 (MPa)	Z_1 (MPa)	m	n
20.0	72.4	0.33	10,000	340	435	300	10.0
148.9	69.3	0.33	10,000	340	435	300	7.0
204.4	65.7	0.33	10,000	340	435	300	4.0
260.0	58.4	0.33	10,000	340	435	300	1.6
371.1	41.5	0.33	10,000	340	435	300	0.6

In the elastic region: isotropic material with Young's modulus E and Poisson's ratio ν; in the plastic region: isotropic work hardening material.

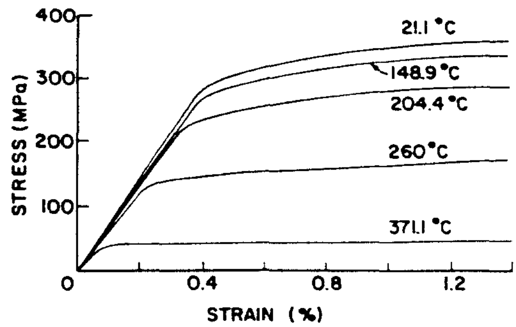

Fig. 1. Stress–strain curves in simple tension at several temperatures of an aluminum alloy (2024-T4) as characterized in Table 1, with an applied strain rate of 0.01 s^{-1}.

Let us consider a square symmetric cross-ply, $[0/90]_s$, laminated plate under simply supported boundary conditions. Within CPT, the buckling condition of this laminate is given by (40). In Fig. 2, the critical stress load, $\hat{\sigma}_{cr}$, for plastic buckling of the present metal matrix laminated plate at a temperature of 204.4°C is shown against the length-to-thickness ratio a/h for compressive strain-rates $\dot{\eta}_{xx} = 10^{-2}\ s^{-1}$ and $\dot{\eta}_{xx} = 10^{-6}\ s^{-1}$. It is clearly seen that the effect of the material rate sensitivity on plastic buckling of the plate is weak in the present case. Also shown in the figure is the graph of the buckling load of the plate when it is assumed to be perfectly elastic (i.e., when the inelastic effects of the aluminum matrix are disregarded). This reveals that the plastic effects of the metal matrix decrease the ability of the plate to sustain loading without buckling. The decrease of the buckling load with increase of a/h should be noted. Furthermore, for a sufficiently thin plate the plastic buckling level approaches the corresponding elastic one. This result is expected since a thin plate buckles at relatively low load levels before appreciable plastic effects develop.

Consider next an anti-symmetrically laminated, angle-ply plate subjected to boundary conditions (26), (28), and (36) which correspond to HSDT and CPT, respectively. The corresponding buckling conditions are determined by using eqns (33) and (38). In Fig. 3, a comparison between the buckling loads predicted by CPT and HSDT of a two-layered $[\pm 30]$ angle-ply square laminated plate at 204.4°C is shown for an applied strain rate of 10^{-2}/s. The figure also includes the corresponding buckling loads of a perfectly elastic plate. It is readily seen that in the present situation the buckling loads obtained by employing CPT and HSDT are close in both cases of inelastic and perfectly elastic plates. For smaller values of a/h it is expected that the buckling loads of the plate obtained by the two theories would be appreciable. For example, assuming $a/h = 5$, the buckling load obtained via CPT is about 1.3 times the corresponding buckling load level obtained from HSDT when the

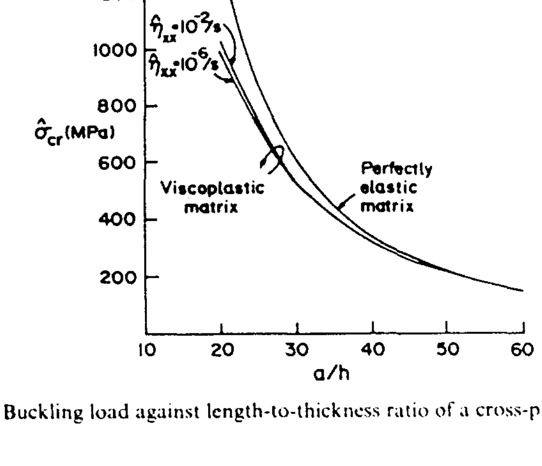

Fig. 2. Buckling load against length-to-thickness ratio of a cross-ply laminate.

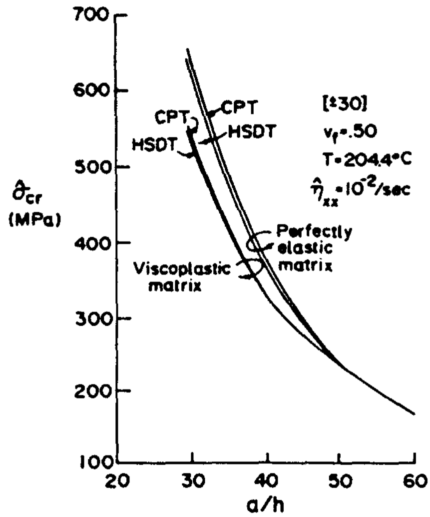

Fig. 3. Buckling load against length-to-thickness ratio of an angle-ply laminate.

plate is assumed to be perfectly elastic. It is not possible, on the other hand, to predict plastic buckling of the plate for such a low value of a/h in the framework of small strain theory considered in the present investigation. The figure demonstrates again the importance of inclusion of the plastic effects, except for thin plates where buckling of the plate already occurs in the elastic region.

In order to study the effect of temperature on the plastic buckling of laminated plates, let us consider square angle-ply laminates at $T = 20°C$ and $T = 204.4°C$. In Fig. 4a, the plastic buckling loads of several types of two-layered angle-ply square laminates are shown at these temperatures. These buckling loads, normalized with respect to the corresponding elastic buckling load (i.e., when the metal matrix is assumed to behave as an elastic material), are shown in Fig. 4b. It is readily seen that at the elevated temperature, the ability of the plate to support loading is reduced. The amount of this reduction depends on the value of

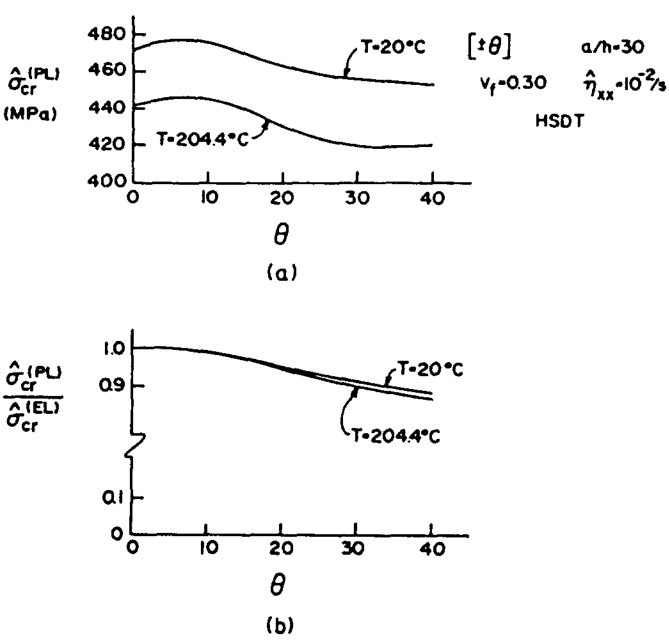

Fig. 4. (a) Critical load against angle of lamination of an angle-ply laminate. (b) Critical load, normalized with respect to the corresponding elastic buckling stress, against angle of lamination of an angle-ply laminate.

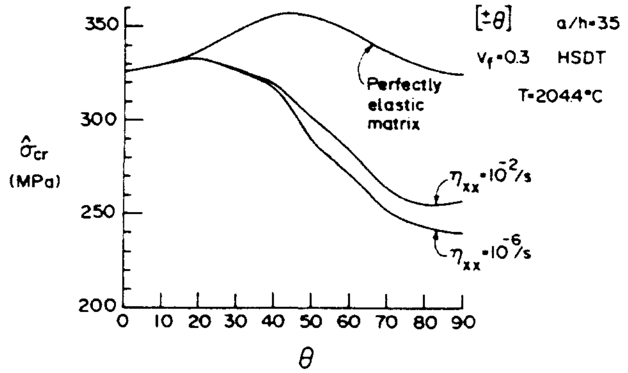

Fig. 5. Buckling load against angle of lamination of an angle-ply laminate, for different values of applied strain-rate.

the lamination angle θ. Furthermore, the effect of taking into account the plastic behavior of the metal matrix on the buckling load of the plate depends on θ as well. The normalized buckling load, however, appears to be slightly dependent on the temperature. The implication is that the temperature reduces the elastic and plastic buckling loads, in the present cases, by almost the same proportion.

Let us consider the elastic and plastic buckling of an anti-symmetric two-layered $[\pm\theta]$ laminated plate predicted by using HSDT. Figure 5 exhibits the variation of the resulting critical loads against θ. By a comparison with the perfectly elastic case, the figure shows that plasticity significantly decreases the buckling loads. Due to the rate sensitivity of the aluminum matrix at 204.4°C, this decrease depends on the applied strain-rate. For a slowly loaded inelastic plate, the buckling point is lower than that obtained by applying a higher strain-rate. For small lamination angles θ, however, the buckling of the plate is approximately elastic and therefore it is not affected by the rate of the applied loading. One can note that whereas the elastic buckling exhibits, in the present fiber/matrix system, a symmetry with respect to $\theta = 45°$, this symmetry is completely lost in the inelastic case. It is interesting to record the total strains of the angle-ply plates at the instant of buckling. In Table 2 the buckling stresses and strains, predicted by employing HSDT, of the inelastic plates $[\pm\theta]$ are given for two values of applied strain rates: $\dot{\eta}_{xx} = 10^{-2}\ s^{-1}$ and $10^{-6}\ s^{-1}$. It is readily seen that appreciable strains develop before buckling takes place for angle-ply plates with lamination angle θ at the vicinity of 50°. It should be noted that the buckling strains are significantly higher for the low loading rate of angle-ply plates with lamination angles in this region. This result is expected, since loading of an inelastic plate at a low strain-rate up to buckling is associated with appreciable plastic flow of the viscoplastic metal matrix.

Table 2. Stresses and strains at the instant of buckling, predicted via HSDT, of $[\pm\theta]$ angle-ply plates, for two values of applied strain rates ($a/h = 35$, $v_f = 0.3$, $T = 204.4°C$)

	Strain-rate $\dot{\eta}_{xx} = 10^{-2}\ s^{-1}$		Strain-rate $\dot{\eta}_{xx} = 10^{-6}\ s^{-1}$	
θ (deg)	Buckling stress (MPa)	Buckling strain (%)	Buckling stress (MPa)	Buckling strain (%)
0	−324.9	−0.19	−324.9	−0.19
10	−327.5	−0.20	−327.5	−0.20
20	−335.4	−0.23	−335.4	−0.23
30	−326.1	−0.28	−325.7	−0.29
40	−320.6	−0.46	−318.7	−0.55
45	−317.8	−0.81	−308.7	−1.13
50	−301.4	−1.10	−287.8	−2.59
60	−284.5	−0.64	−272.9	−0.75
70	−262.9	−0.36	−251.5	−0.38
80	−254.1	−0.29	−241.2	−0.28
90	−255.9	−0.28	−239.5	−0.26

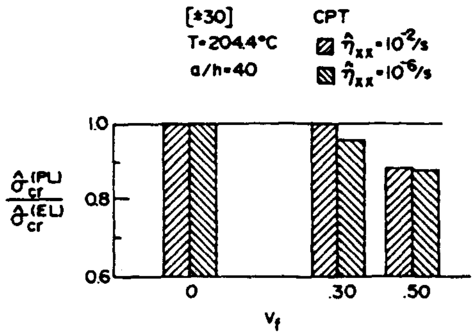

Fig. 6. Critical load, normalized with respect to the corresponding elastic buckling stress, against fiber volume ratio.

It is possible to study the effect of different values of fiber volume fraction v_f on the plastic buckling of a metal matrix composite plate. To this end, consider a [$\pm$30] angle-ply laminated plate with $a/h = 40$, at a temperature of 204.4°C. In Fig. 6 results for the plastic buckling of the plate using CPT are shown for three values of v_f: 0, 0.3 and 0.5. Here, the buckling levels are normalized with respect to the corresponding buckling loads of the perfectly elastic plate. Plastic buckling loads for two different loading rates are shown in the figure: $10^{-2}\ s^{-1}$ and $10^{-6}\ s^{-1}$. The case of $v_f = 0$ corresponds to a homogeneous plate which buckles at the present value of $a/h = 40$ in the elastic region. As the reinforcement volume ratio increases, the buckling of the plate takes place in the plastic region. This is due to the increasing effect of the fibers, as a result of which the plate can sustain higher loading. These high load levels lead to yielding and plastic flow of the metal matrix. Since for $v_f = 0$ buckling occurs in the elastic region far away from the yield point, the effect of different load rate levels is absent. For moderate values of v_f this effect is pronounced due to the existence of plastic flow. For higher values of v_f the effect of fibers is dominant and the rate sensitivity of the composite plate is weak.

Let us study the effect of number of plies on the plastic buckling load level of a $[\pm\theta]_k$ angle-ply composite plate. To this end, consider a $[\pm 30]_k$ with $k = 1, 2, 3$. The resulting plastic buckling loads obtained within CPT and HSDT are shown in Fig. 7. It is readily observed that, as in the perfectly elastic case, the plastic buckling increases with increasing number of plies of the laminated plate. It can be seen that HSDT predicts slightly lower buckling loads as compared with CPT. The small differences between HSDT and CPT predictions in the present situation are attributed, as previously mentioned, to the relatively high value of length-to-thickness ratio of the laminated plate ($a/h = 30$).

5. CONCLUSIONS

By employing a micromechanical analysis which can provide the instantaneous properties of metal matrix composites, the plastic buckling of laminated plates is determined. It is shown that in the determination of the critical loading of the plate, the satisfaction of the

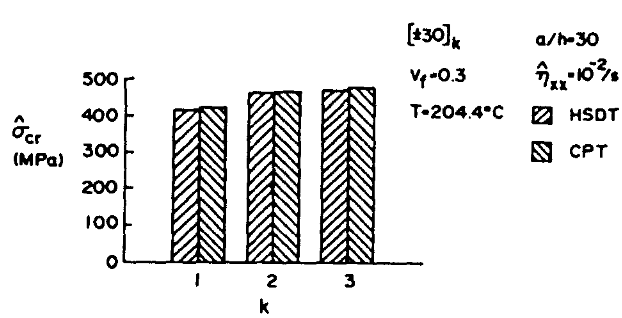

Fig. 7. Buckling load against number of layers of an angle-ply laminate.

corresponding buckling condition has to be checked at all stages of loading history. The method is illustrated for the plastic buckling analysis of boron aluminum composite plates under various situations. The effect of the elastic–viscoplastic behavior of the aluminum matrix and its rate sensitivity at elevated temperatures on the plastic buckling of the plates is presented.

REFERENCES

Aboudi, J. (1989). Micromechanical analysis of composites by the method of cells. *Appl. Mech. Rev.* **42**, 193–221.

Accorci, M. L. and Nemat-Nasser, S. (1986). Bounds on the overall elastic and instantaneous elasto-plastic moduli of periodic composites. *Mech. Mater.* **5**, 209–220.

Bijlaard, P. P. (1949). Theory and tests on the plastic stability of plates and shells. *J. Aeronautical Sci.* **42**, 193–221.

Bodner, S. R. and Partom, Y. (1975). Constitutive equations for elastic–viscoplastic strain-hardening materials. *J. Appl. Mech.* **42**, 385–389.

Bodner, S. R., Naveh, M. and Merzer, A. M. (1991). Deformation and buckling of axisymmetric viscoplastic shells under thermomechanical loading. *Int. J. Solids Structures* **27**, 1915–1924.

Bushnell, D. (1982). Plastic buckling. In *Pressure Vessels and Piping: Design Technology, A Decade of Progress* (Edited by S. Y. Zamrik and D. Dietrich), pp. 47–117. ASME, New York.

Hendelman, G. H. and Prager, W. (1948). Plastic buckling of a rectangular plate under edge thrusts. NACA TN 1530.

Hill, R. (1958). A general theory of uniqueness and stability in elastic-plastic solids. *J. Mech. Phys. Solids* **6**, 236–249.

Hutchinson, J. W. (1974). Plastic buckling. In *Advances in Applied Mechanics*, Vol. 14, pp. 529–541. Academic Press, New York.

Kapania, R. K. and Raciti, S. (1989). Recent advances in analysis of laminated beams and plates. Part I: Shear effects and buckling. *AIAA Jl* **27**, 923–934.

Leissa, A. W. (1985). Buckling of laminated composite plates and shell panels. Flight Dynamic Laboratory, Wright Patterson Air Force Base, Dayton, Ohio, Report AFWAL-TR-85-3069.

Leissa, A. W. (1987). A review of laminated composite plate buckling. *Appl. Mech. Rev.* **40**, 575–591.

Paley, M. and Aboudi, J. (1991a). Overall instantaneous properties of metal matrix composites. *Composite Sci. Tech.* **41**, 411–429.

Paley, M. and Aboudi, J. (1991b). Viscoplastic bifurcation buckling of plates. *AIAA Jl* **29**, 627–632.

Reddy, J. N. and Phan, N. D. (1985). Stability and vibrations of isotropic, orthotropic and laminated plates according to a higher-order shear deformation theory. *J. Sound Vibr.* **98**, 157–170.

Teply, J. L. and Dvorak, G. J. (1988). Bounds on overall instantaneous properties of elastic–plastic composites. *J. Mech. Phys. Solids* **36**, 29–58.

Tvergaard, V. (1985). Rate-sensitivity in elastic plastic panel buckling. In *Aspects of the Analysis of Plate Structures, A Volume in Honor of W. H. Wittrick* (Edited by D. J. Dawe *et al.*), pp. 293–308. Clarendon Press, Oxford.

Tvergaard, V. (1989). Plasticity and creep at finite strains. *Proc. 17th Congress of Theoretical and Applied Mechanics* (Edited by P. Germain, M. Piau and D. Caillerie). Elsevier, Amsterdam.

Whitney, J. M. (1987). *Structural Analysis of Laminated Anisotropic Plates*. Technomic, Lancaster, PA.

Whitney, J. M. and Leissa, A. W. (1969). Analysis of heterogeneous anisotropic plates. *J. Appl. Mech.* **36**, 261–266.